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Existence of a Unique Weak Solution to a Nonlinear Non-Autonomous Time-Fractional Wave Equation (of Distributed-Order)

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Abstract: We study an initial-boundary value problem for a fractional wave equation of time distributed-order with a nonlinear source term. The coefficients of the second order differential operator are dependent on the spatial and time variables. We show the existence of a unique weak solution to the problem under low regularity assumptions on the data, which includes weakly singular solutions in the class of admissible problems. A similar result holds true for the fractional wave equation with Caputo fractional derivative.

Keywords: time-fractional wave equation; distributed-order; non-autonomous; time discretization; existence; uniqueness

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1. Introduction

We consider a bounded domain $\Omega \subset \mathbb{R}^d$ $(d \in \mathbb{N})$ with a Lipschitz continuous boundary $\partial\Omega$. We denote the final time by T, and we define $Q_T := \Omega \times (0, T]$ and $\Sigma_T := \partial\Omega \times (0, T]$. The objective of this paper is to show the existence of a unique u for given f, $\tilde{u}_0$ and $\tilde{v}_0$ to the fractional wave equation of time distributed-order (DO) with nonlinear source term given by

$$\begin{cases} \left(\mathcal{D}_t^{(\mu)} u\right)(\mathbf{x}, t) + L(\mathbf{x}, t)u(\mathbf{x}, t) &= f(\mathbf{x}, t) + F\left(u(\mathbf{x}, t)\right) \qquad (\mathbf{x}, t) \in Q_T, \\ u(\mathbf{x}, t) &= 0 \qquad (\mathbf{x}, t) \in \Sigma_T, \\ u(\mathbf{x}, 0) &= \tilde{u}_0(\mathbf{x}) \qquad \mathbf{x} \in \Omega, \\ \partial_t u(\mathbf{x}, 0) &= \tilde{v}_0(\mathbf{x}) \qquad \mathbf{x} \in \Omega. \end{cases} \tag{1}$$

$\mathcal{D}_t^{(\mu)} u$ stands for the time DO fractional derivative defined by

$$\left(\mathcal{D}_t^{(\mu)} u\right)(\mathbf{x}, t) = \int_1^2 \left(\partial_t^\beta u\right)(\mathbf{x}, t)\mu(\beta)\,\mathrm{d}\beta, \qquad (\mathbf{x}, t) \in Q_T, \tag{2}$$

with weight function $\mu : [1, 2] \to \mathbb{R}$ satisfying

$$\mu \in \mathrm{L}^1(1, 2), \quad \mu \geqslant 0, \quad \mu \not\equiv 0,$$

and with $\partial_t^\beta u$ the Caputo derivative of order $\beta \in (1, 2)$ defined by [1,2]

$$\left(\partial_t^\beta u\right)(\mathbf{x}, t) = \int_0^t \frac{(t-s)^{1-\beta}}{\Gamma(2-\beta)} \partial_{ss} u(\mathbf{x}, s)\ \mathrm{d}s, \qquad (\mathbf{x}, t) \in Q_T,$$

where Γ denotes the Gamma function. In this paper, we rewrite the DO fractional derivative defined in (2) as

$$\left(\mathcal{D}_t^{(\mu)} u\right)(\mathbf{x}, t) = (k * \partial_{tt} u)\,(\mathbf{x}, t), \qquad (\mathbf{x}, t) \in Q_T,$$

where

$$k(t) = \int_1^2 \frac{t^{1-\beta}}{\Gamma(2-\beta)} \mu(\beta)\,\mathrm{d}\beta,$$

and the symbol '$*$' stands for the convolution product defined by $(k * z)\,(t) = \int_0^t k(t-s)z(s)\,\mathrm{d}s$. The second-order linear differential operator L is defined as follows

$$L(\mathbf{x}, t)u(\mathbf{x}, t) = -\nabla \cdot (\boldsymbol{A}(\mathbf{x}, t)\nabla u(\mathbf{x}, t) + \boldsymbol{b}(\mathbf{x}, t)u(\mathbf{x}, t)) + c(\mathbf{x}, t)u(\mathbf{x}, t), \tag{3}$$

where $((\mathbf{x}, t) \in Q_T)$

$$\boldsymbol{A}(\mathbf{x}, t) = \left(a_{i,j}(\mathbf{x}, t)\right)_{i,j=1,\ldots,d}, \quad \boldsymbol{b}(\mathbf{x}, t) = (b_1(\mathbf{x}, t), b_2(\mathbf{x}, t), \ldots, b_d(\mathbf{x}, t)).$$

The system (1) can be used to model the propagation of mechanical waves in viscoelastic materials [3–9]. We mention the most important results available in literature related to the fractional wave equations, which are also connected to the subject of this paper. The existence and uniqueness of a solution to autonomous (time-independent elliptic part) constant-order fractional wave equations is studied in References [10,11]. A fundamental solution to the fractional wave equation of constant-order is determined in Reference [12] and to the Cauchy problem for the 1D DO diffusion-wave equation in Reference [13]. In Reference [14], a Cauchy problem for a time-fractional DO multi-dimensional diffusion-wave equation is investigated in the space of tempered distributions. The unique solvability of the Cauchy problem for inhomogeneous DO differential equations in a Banach space with a linear bounded operator in the right-hand side is studied in Reference [15]. In Reference [16], the authors give existence and uniqueness of weak solutions results together with energy estimates for problem (1) (see Theorem 3.2) considering instead of L the fractional powers of order $s \in (0, 1)$ of a self-adjoint ($\boldsymbol{A}^{\mathsf{T}} = \boldsymbol{A}$ and $\boldsymbol{b} = \mathbf{0}$) and uniformly elliptic second order operator with time-independent coefficients (the domain Ω is an open, bounded, and convex subset of $\mathbb{R}^d$). Their approach is based on a Galerkin technique (the solution has a representation formula separated in the independent variables), spectral theory and the Mittag-Leffler analysis. However, this type of analysis is not appropriate for analyzing problem (1) as the coefficients in the governing operator are allowed to be time dependent. To the best of our knowledge, no paper deals with the existence of a unique weak solution to problem (1), which is the main goal of this paper. Now, we discuss numerical methods for solving the (DO) fractional wave equations as the method that will be applied in this article also includes a time-discrete scheme for computations. A fully discrete difference Crank-Nicolson scheme is derived for a diffusion-wave system in Reference [17]. A difference scheme for a one-dimensional DO time-fractional wave equation is derived and analyzed [18]. Two alternating direction implicit difference schemes for solving the two-dimensional time DO wave equations were developed in Reference [19]. The element-free Galerkin method is used in Reference [20] to solve the two-dimensional DO time-fractional diffusion-wave equation. Finite difference schemes for a multidimensional time-fractional wave equation of DO with a nonlinear source term are studied in Reference [21]. Two temporal second-order schemes are derived and analyzed for the time multi-term fractional diffusion-wave equation based on the order reduction technique in Reference [22]. A fast and linearized finite difference method to solve a nonlinear time-fractional wave equation with multi fractional orders is studied in Reference [23]. In all these articles it is usually assumed that the solution $u(t)$ (ignoring the space variable) lies in $\mathrm{C}^2([0, T])$ or $\mathrm{C}^3([0, T])$, but it is well-known that $\partial_{tt} u$ blows up as $t \to 0^+$ in some practical situations [16,24,25]. An overview of numerical methods for time-fractional evolution equations with nonsmooth data is given in Reference [26].

However, in this contribution, we follow a standard procedure for showing the existence of a weak solution (time-discretization method), which is recently applied in Reference [27] for a non-autonomous time fractional diffusion equation of DO. We first state this result below, see (Reference [27] Theorem 3.1).

Theorem 1 (Fractional diffusion equation (of DO)). *Consider* $\left(\partial_t^\beta u\right)(t) = (k * \partial_t u)(t) = \int_0^t \frac{(t-s)^{-\beta}}{\Gamma(1-\beta)} \partial_s u(s)\, \mathrm{d}s$ *with* $\beta \in (0,1)$ *or* $\left(\mathcal{D}_t^{(\mu)} u\right)(t) = (k * \partial_t u)(t) = \int_0^1 \left(\partial_t^\beta u\right)(t) \mu(\beta)\, \mathrm{d}\beta$ *(i.e., DO) in problem* (1) *with* $F \equiv 0$*. Assume that*

- $\mu \in \mathrm{L}^1(0,1), \mu \geqslant 0, \mu \not\equiv 0$ *(if DO);*
- $\tilde{u}_0 \in \mathrm{H}_0^1(\Omega)$*;*
- $f \in \mathrm{H}^1\left((0,T), \mathrm{H}_0^1(\Omega)^*\right)$*, or* $f \in \mathrm{L}^2\left((0,T), \mathrm{H}_0^1(\Omega)^*\right)$ *with* $\|\partial_t f(t)\|_{\mathrm{H}_0^1(\Omega)^*} \leqslant Ct^{-\alpha}$ *for all* $t \in (0,T]$ *and some constant* $\alpha \in (0,1)$*;*
- $\boldsymbol{A} \in \left(\mathrm{L}^\infty\left(\overline{Q_T}\right)\right)^{d\times d}$ *is uniformly elliptic with* $\boldsymbol{A}^\top = \boldsymbol{A}$ *and* $\partial_t \boldsymbol{A} \in \left(\mathrm{L}^\infty\left(\overline{Q_T}\right)\right)^{d\times d}$*;*
- $\boldsymbol{b} = \boldsymbol{0}$*;*
- $c \in \mathrm{L}^\infty\left(\overline{Q_T}\right)$ *with* $c \geqslant 0$ *in* $\overline{Q_T}$*, and* $\partial_t c \in \mathrm{L}^\infty\left(\overline{Q_T}\right)$*.*

Then, there exists a unique weak solution u *to the problem* (1) *with* $u \in \mathrm{C}\left([0,T], \mathrm{H}_0^1(\Omega)^*\right) \cap \mathrm{L}^\infty\left((0,T), \mathrm{H}_0^1(\Omega)\right)$ *and* $k * \partial_t u \in \mathrm{L}^2\left((0,T), \mathrm{H}_0^1(\Omega)^*\right)$*.*

We will show that for the fractional wave equation (of DO) the analysis goes through without information about $\partial_t f$ and when $\boldsymbol{b} \neq \boldsymbol{0}$ (i.e., non-symmetric L). In Section 2, we define the weak formulation of (1). Afterwards, in Section 3, we establish the existence of a unique solution to the variational problem by applying Rothe's method, which is the most general result in literature concerning the well-posedness of the non-autonomous time-fractional wave equation. The obtained regularity on u takes into account the possible singularity at $t = 0$ in case of non-smooth solutions. We finally note that when showing the existence and uniqueness of a solution, we will use Proposition 6 and 10 from Reference [28], which can be used independently of the type of operator (diffusion/wave).

Remark 1 (Additional notations). *We denote by* $(\cdot,\cdot)$ *the standard inner product in* $\mathrm{L}^2(\Omega)$ *and by* $\|\cdot\|$ *its induced norm. The space* $\mathrm{L}^2(\partial\Omega)$ *on the boundary is defined analogously as the space* $\mathrm{L}^2(\Omega)$*. The multi-index* $\boldsymbol{\alpha}$ *is a d-dimensional vector with* $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d)$*,* $\alpha_i \geq 0$ *with* $\alpha_i \in \mathbb{Z}$*. The length of* $\boldsymbol{\alpha}$ *is given by* $|\boldsymbol{\alpha}|_1 = \sum_{i=1}^d \alpha_i$*. The function* $u \in \mathrm{L}^2(\Omega)$ *is an element of* $\mathrm{H}^k(\Omega)$ *if all generalized (weak) derivatives* $D^{\boldsymbol{\alpha}} u$ *of* u *up to order* k *exist and belong to* $\mathrm{L}^2(\Omega)$*, that is,*

$$\mathrm{H}^k(\Omega) = \left\{u \in \mathrm{L}^2(\Omega) : D^{\boldsymbol{\alpha}} u \in \mathrm{L}^2(\Omega), |\boldsymbol{\alpha}|_1 \leqslant k\right\}.$$

The space $\mathrm{H}^k(\Omega)$ *is called a Sobolev space of order* k *and is equipped with the norm*

$$\|u\|_{\mathrm{H}^k(\Omega)} = \left(\sum_{|\boldsymbol{\alpha}|_1 \leqslant k} \left\|D^{\boldsymbol{\alpha}} u\right\|_{\mathrm{L}^2(\Omega)}^2\right)^{\frac{1}{2}}.$$

The function space $\mathrm{H}_0^1(\Omega)$ *is defined as*

$$\mathrm{H}_0^1(\Omega) := \{u \in \mathrm{H}^1(\Omega) : \gamma(u) = 0\} = \{u \in \mathrm{H}^1(\Omega) : u = 0 \text{ on } \partial\Omega\},$$

were γ *denotes the trace map from* $\mathrm{H}^1(\Omega)$ *to* $\mathrm{L}^2(\partial\Omega)$*.*

Next, we consider an abstract Banach space X *with norm* $\|\cdot\|_X$ *and* $p \in [1,\infty)$*. The space* $\mathrm{L}^p((0,T),X)$ *consists of measurable functions* $u : (0,T) \to X$ *such that*

$$\|u\|_{\mathrm{L}^p((0,T),X)} = \left(\int_0^T \|u(t)\|_X^p \,\mathrm{d}t\right)^{1/p} < \infty.$$

The space $\mathrm{C}\left([0,T],X\right)$ *consists of continuous functions* $u:[0,T]\to X$ *satisfying*

$$\|u\|_{\mathrm{C}([0,T],X)} = \max_{t\in[0,T]} \left\|u(t)\right\|_X < \infty.$$

The space $\mathrm{L}^\infty\left((0,T),X\right)$ *consists of all measurable functions* $u:(0,T)\to X$ *that are essentially bounded. The space* $\mathrm{H}^1((0,T),X)$ *consists of functions* $u:(0,T)\to X$ *such that the weak derivative* u' *exists and*

$$\|u\|_{\mathrm{H}^1((0,T),X)} = \left[\int_0^T \left(\left\|u(t)\right\|_X^2 + \left\|u'(t)\right\|_X^2\right)\mathrm{d}t\right]^{\frac{1}{2}} < \infty.$$

The values C, ε *and* C_ε *are considered to be generic and positive constants (independent of the discretization parameter) throughout the paper, where* ε *is arbitrarily small and* C_ε *arbitrarily large, that is,* $C_\varepsilon = C\left(1+\varepsilon+\frac{1}{\varepsilon}\right)$. *The same notation for different constants is used, but the meaning should be clear from the context.*

2. Weak Formulation

First, we define the variational formulation of problem (1) as follows: search $u\in \mathrm{L}^2\left((0,T),\mathrm{H}_0^1(\Omega)\right)$ with $k*\partial_{tt}u \in \mathrm{L}^2\left((0,T),\mathrm{H}_0^1(\Omega)^*\right)$ such that for almost all (a.a.) $t\in(0,T)$ and for all $\varphi\in\mathrm{H}_0^1(\Omega)$ it holds that

$$\langle (k*\partial_{tt}u)\,(t),\varphi\rangle_{\mathrm{H}_0^1(\Omega)^*\times\mathrm{H}_0^1(\Omega)} + \mathcal{L}(t)\,(u(t),\varphi) = (f,\varphi) + (F(u(t)),\varphi)\,, \tag{4}$$

where a bilinear form $\mathcal{L}$ is associated with the differential operator L defined in (3) as follows

$$\begin{aligned}\mathcal{L}(t)\,(u(t),\varphi) &:= (L(t)u(t),\varphi)\\ &= (\boldsymbol{A}(t)\nabla u(t)+\boldsymbol{b}(t)u(t),\nabla\varphi) + (c(t)u(t),\varphi)\,, \quad \text{with } u(t),\varphi\in\mathrm{H}_0^1(\Omega).\end{aligned}$$

For completeness, we summarize the properties of the singular kernel k below

- k is strongly positive definite since $k(t)\geqslant 0$ for all $t>0$, $\partial_t k(t)\leqslant 0$ for all $t>0$ and $\partial_{tt}k(t)\geqslant 0$ for all $t>0$ [29];
- $k\in\mathrm{L}^1(0,T)$ since

$$\int_0^T \left|k(t)\right|\mathrm{d}t = \int_1^2 \frac{\mu(\beta)}{\Gamma(2-\beta)}\left(\int_0^T t^{1-\beta}\right)\mathrm{d}t\,\mathrm{d}\beta = \int_1^2 \frac{\mu(\beta)}{\Gamma(3-\beta)}T^{2-\beta}\,\mathrm{d}\beta \leqslant 2\overline{\mu}\max\{1,T\}, \tag{5}$$

  using that $\Gamma(z)\geqslant\frac{1}{2}$ for $z\in(1,2)$ and $\overline{\mu} := \int_1^2 \mu(\beta)\,\mathrm{d}\beta$;
- $\partial_t k\in\mathrm{L}^1_{\mathrm{loc}}(0,T)$ for any $[t_1,t_2]\subset(0,T)$ since

$$\int_{t_1}^{t_2}\left|\partial_t k(t)\right|\mathrm{d}t = \int_1^2 \frac{\mu(\beta)}{\Gamma(2-\beta)}\left(t_1^{1-\beta}-t_2^{1-\beta}\right)\mathrm{d}\beta \leqslant 2\overline{\mu}\max\{1,t_1^{-1}\},$$

  using that $\Gamma(z)\geqslant\Gamma(1)=1$ for $z\in(0,1)$.

We state the following assumptions on the data functions in (3), which will be used throughout the paper. The matrix $\boldsymbol{A}=\left(a_{ij}(\mathbf{x},t)\right)$ is a $d\times d$ matrix-valued function such that $\boldsymbol{A}\in\left(\mathrm{L}^\infty\left(\overline{Q_T}\right)\right)^{d\times d}$ is uniformly elliptic, that is, there exists a strict positive constant α such that

$$\sum_{i,j=1}^d a_{ij}(\mathbf{x},t)\xi_i\xi_j \geqslant \alpha\,|\boldsymbol{\xi}|^2\,, \quad \text{for a.a. } (\mathbf{x},t)\in\overline{Q_T} \text{ and for all } \boldsymbol{\xi}\in\mathbb{R}^d.$$

We suppose that $\boldsymbol{b} \in \mathbf{L}^\infty\left(\overline{Q_T}\right)$ and $c \in \mathrm{L}^\infty\left(\overline{Q_T}\right)$ such that

$$c(\mathbf{x},t) - \frac{\|\boldsymbol{b}\|^2_{\mathbf{L}^\infty(\overline{Q_T})}}{2\alpha} \geqslant 0, \quad (\mathbf{x},t) \in \overline{Q_T}.$$

Moreover, we suppose that

- $\boldsymbol{A}^\top = \boldsymbol{A}$;
- $\partial_t \boldsymbol{A} \in \left(\mathrm{L}^\infty\left(\overline{Q_T}\right)\right)^{d\times d}$;
- $(\nabla \cdot \boldsymbol{b})(t) \in \mathrm{L}^\infty(\Omega)$ for all $t \in (0,T)$;
- $\partial_t \boldsymbol{b} \in \left(\mathrm{L}^\infty\left(\overline{Q_T}\right)\right)^d$.

Therefore, we have that

$$\begin{aligned} &\mathcal{L}(t)\,(u,\varphi) \leqslant C\,\|u\|_{\mathrm{H}^1_0(\Omega)}\left\|\varphi\right\|_{\mathrm{H}^1_0(\Omega)}, && \forall u,\varphi \in \mathrm{H}^1_0(\Omega); \\ &\mathcal{L}(t)\,(\varphi,\varphi) \geqslant \frac{\alpha}{2}\left\|\nabla\varphi\right\|^2, && \forall \varphi \in \mathrm{H}^1_0(\Omega). \end{aligned} \tag{6}$$

It follows from the Friedrichs inequality that the bilinear form $\mathcal{L}$ is $\mathrm{H}^1_0(\Omega)$-elliptic. In the next section, we study the existence and uniqueness of a solution.

3. Existence of a Solution

We employ a semidiscretization in time based on Rothe's method to address the existence of a weak solution to (1). First, we discretize the time interval $[0,T]$ into $n \in \mathbb{N}$ equidistant subintervals $[t_{i-1}, t_i]$ with uniform time step $\tau = \frac{T}{n} < 1$, that is, $t_i = i\tau, i = 0,\ldots,n$. We denote the approximation of u at time $t = t_i$ $(0 \leqslant i \leqslant n)$ by u_i, and we approximate the first and second order time derivative at time $t = t_i$ by the backward Euler finite-difference formulas

$$\partial_t u(t_i) \approx \delta u_i = \frac{u_i - u_{i-1}}{\tau}, \quad \partial_{tt} u(t_i) \approx \delta^2 u_i = \frac{\delta u_i - \delta u_{i-1}}{\tau} = \frac{u_i - u_{i-1}}{\tau^2} - \frac{\delta u_{i-1}}{\tau}, \quad 1 \leqslant i \leqslant n.$$

These notations are also used for any function $z \neq u$. Next, we define the time discrete convolution as follows

$$(k * z)(t_i) \approx (k * z)_i := \sum_{l=1}^{i} k_{i+1-l} z_l \tau, \tag{7}$$

with

$$(k * z)_0 := 0.$$

We approximate problem (4) at time $t = t_i$ by: Find $u_i \in \mathrm{H}^1_0(\Omega)$, $i = 1,2,\ldots,n$, such that

$$\langle (k * \delta^2 u)_i, \varphi \rangle_{\mathrm{H}^1_0(\Omega)^* \times \mathrm{H}^1_0(\Omega)} + \mathcal{L}_i(u_i,\varphi) = (f_i,\varphi) + (F(u_{i-1}),\varphi), \quad \forall \varphi \in \mathrm{H}^1_0(\Omega). \tag{8}$$

Employing the time discrete convolution (7), the discrete problem can be equivalently written as

$$a_i(u_i,\varphi) = \langle \tilde{F}_i, \varphi \rangle, \quad \forall \varphi \in \mathrm{H}^1_0(\Omega), \tag{9}$$

with

$$a_i(u,\varphi) := \frac{k(\tau)}{\tau}\,(u,\varphi) + \mathcal{L}_i(u,\varphi),$$

and

$$\langle \tilde{F}_i, \varphi \rangle := (f_i,\varphi) + (F(u_{i-1}),\varphi) + \frac{k(\tau)}{\tau}\,(u_{i-1},\varphi) + k(\tau)\,(\delta u_{i-1},\varphi) - \sum_{l=1}^{i-1} k_{i+1-l}\left(\delta^2 u_l, \varphi\right)\tau.$$

The well-posedness of this problem under appropriate assumptions on the initial data follows inductively from the Lax-Milgram lemma and it is stated in the following lemma.

Lemma 1. *Suppose that $\tilde{u}_0 \in \mathrm{L}^2(\Omega)$, $\tilde{v}_0 \in \mathrm{L}^2(\Omega)$ and $f \in \mathrm{L}^\infty\left((0,T),\mathrm{L}^2(\Omega)\right)$. Moreover, assume that the nonlinear source term $F : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous, that is,*

$$\left|F(s_1) - F(s_2)\right| \leqslant L_F\,|s_1 - s_2|\,, \quad s_i \in \mathbb{R},$$

where L_F is a positive constant. Then, for any $i = 1, 2, \ldots, n$, there exists a unique $u_i \in \mathrm{H}^1_0(\Omega)$ solving (8).

3.1. A Priori Estimates

In this subsection, we derive the a priori estimates that we require to be able to show the existence of a solution. Consider the evolution triple $\mathrm{H}^1_0(\Omega) \subset \mathrm{L}^2(\Omega) \cong (\mathrm{L}^2(\Omega))^* \subset \mathrm{H}^1_0(\Omega)^*$. Then, from ([30], Lemma 3.2), it follows for a sequence $(z_i)_{i\in\mathbb{N}}$ in $\mathrm{H}^1_0(\Omega)$ that

$$\sum_{i=1}^{j} \langle \delta(k*z)_i, z_i\rangle_{\mathrm{H}^1_0(\Omega)^*\times\mathrm{H}^1_0(\Omega)}\tau = \sum_{i=1}^{j} \left(\delta(k*z)_i, z_i\right)\tau \geqslant \frac{1}{2}\left(k*\|z\|^2\right)_j + \frac{1}{2}\sum_{i=1}^{j} k_i\,\|z_i\|^2\,\tau, \quad j \in \mathbb{N}, \tag{10}$$

with

$$\left(k*\|z\|^2\right)_j := \sum_{l=1}^{j} k_{j+1-l}\,\|z_l\|^2\,\tau.$$

Lemma 2. *Let the assumptions of Lemma 1 be fulfilled. Moreover, assume that $\tilde{u}_0 \in \mathrm{H}^1_0(\Omega)$. Then, positive constants C and τ_0 exist such that for any $\tau < \tau_0$ and for every $j = 1, 2, \ldots, n$, the following relation holds*

$$\left(k*\|\delta u\|^2\right)_j + \sum_{i=1}^{j} k_i\,\|\delta u_i\|^2\,\tau + \sum_{i=1}^{j} \|\delta u_i\|^2\,\tau + \left\|\nabla u_j\right\|^2 + \sum_{i=1}^{j} \|\nabla u_i - \nabla u_{i-1}\|^2 \leqslant C.$$

Proof. We put $\varphi = \delta u_i \tau$ in (8) and sum the result up for $i = 1, \ldots, j$ with $1 \leqslant j \leqslant n$. Employing the following relation for any sequence $\{z_i\}_{i\in\mathbb{N}} \subset \mathrm{L}^2(\Omega)$:

$$\delta(k*z)_i(\mathbf{x}) = k_i(\mathbf{x})z_0(\mathbf{x}) + (k*\delta z)_i(\mathbf{x}), \quad \text{for a.a } \mathbf{x} \in \Omega, \tag{11}$$

we obtain that

$$\sum_{i=1}^{j} \left(\delta(k*\delta u)_i, \delta u_i\right)\tau + \sum_{i=1}^{j} \mathcal{L}_i(u_i, \delta u_i)\tau = \sum_{i=1}^{j} \left(f_i, \delta u_i\right)\tau + \sum_{i=1}^{j} \left(F(u_{i-1}), \delta u_i\right)\tau + \sum_{i=1}^{j} \left(k_i\tilde{v}_0, \delta u_i\right)\tau. \tag{12}$$

For the first term on the LHS, we get that

$$\sum_{i=1}^{j} \left(\delta(k*\delta u)_i, \delta u_i\right)\tau \overset{(10)}{\geqslant} \frac{1}{2}\left(k*\|\delta u\|^2\right)_j + \frac{1}{4}\sum_{i=1}^{j} k_i\,\|\delta u_i\|^2\,\tau + \frac{k(T)}{4}\sum_{i=1}^{j} \|\delta u_i\|^2\,\tau.$$

We use per partes formula for a symmetric bilinear form for the second term on the LHS of (12):

$$\sum_{i=1}^{j} r(t_i; z_i, z_i - z_{i-1}) = \frac{1}{2}r(t_j; z_j, z_j) - \frac{1}{2}r(0; z_0, z_0) + \frac{1}{2}\sum_{i=1}^{j}\left(r(t_i; \delta z_i, \delta z_i)\tau^2 - \delta r(t_i; z_{i-1}, z_{i-1})\tau\right).$$

Therefore, by the symmetry of $\boldsymbol{A}$, we have that

$$\sum_{i=1}^{j} (\boldsymbol{A}_i \nabla u_i, \nabla \delta u_i)\, \tau = \frac{1}{2}\left(\boldsymbol{A}_j \nabla u_j, \nabla u_j\right) - \frac{1}{2}\left(\boldsymbol{A}_0 \nabla \tilde{u}_0, \nabla \tilde{u}_0\right) - \frac{1}{2}\sum_{i=1}^{j} (\delta \boldsymbol{A}_i \nabla u_{i-1}, \nabla u_{i-1})\, \tau + \frac{1}{2}\sum_{i=1}^{j} (\boldsymbol{A}_i (\nabla u_i - \nabla u_{i-1}), \nabla u_i - \nabla u_{i-1}),$$

and thus

$$\sum_{i=1}^{j} (\boldsymbol{A}_i \nabla u_i, \nabla \delta u_i)\, \tau \geqslant \frac{\alpha}{2}\left\|\nabla u_j\right\|^2 - C - C\sum_{i=1}^{j-1} \|\nabla u_i\|^2\, \tau + \frac{\alpha}{2}\sum_{i=1}^{j} \|\nabla u_i - \nabla u_{i-1}\|^2 .$$

Using the ε-Young inequality and the Friedrichs inequality, we get that

$$\left|\sum_{i=1}^{j} (c_i u_i, \delta u_i)\, \tau\right| \leqslant C_{\varepsilon_1} \sum_{i=1}^{j} \|\nabla u_i\|^2\, \tau + \varepsilon_1 \sum_{i=1}^{j} \|\delta u_i\|^2\, \tau,$$

and

$$\left|\sum_{i=1}^{j} (\boldsymbol{b}_i u_i, \nabla \delta u_i)\, \tau = -\sum_{i=1}^{j} (\nabla \cdot (\boldsymbol{b}_i u_i), \delta u_i)\, \tau\right| \leqslant C_{\varepsilon_2} \sum_{i=1}^{j} \|\nabla u_i\|^2\, \tau + \varepsilon_2 \sum_{i=1}^{j} \|\delta u_i\|^2\, \tau.$$

The terms in the RHS of (12) can be estimated as follows

$$\left|\sum_{i=1}^{j} (f_i, \delta u_i)\, \tau\right| \leqslant C_{\varepsilon_3} + \varepsilon_3 \sum_{i=1}^{j} \|\delta u_i\|^2\, \tau,$$

$$\left|\sum_{i=1}^{j} (F(u_{i-1}), \delta u_i)\, \tau\right| \leqslant C_{\varepsilon_4} + C_{\varepsilon_4} \sum_{i=1}^{j-1} \|\nabla u_i\|^2\, \tau + \varepsilon_4 \sum_{i=1}^{j} \|\delta u_i\|^2\, \tau$$

and

$$\left|\sum_{i=1}^{j} (k_i \tilde{v}_0, \delta u_i)\, \tau\right| \leqslant C_{\varepsilon_5} + \varepsilon_5 \sum_{i=1}^{j} k_i\, \|\delta u_i\|^2\, \tau,$$

since $k \in \mathrm{L}^1(0,T)$. Collecting the previous estimates gives

$$\frac{1}{2}\left(k * \|\delta u\|^2\right)_j + \left(\frac{1}{4} - \varepsilon_5\right)\sum_{i=1}^{j} k_i\, \|\delta u_i\|^2\, \tau + \left(\frac{k(T)}{4} - \varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4\right)\sum_{i=1}^{j} \|\delta u_i\|^2\, \tau + \frac{\alpha}{2}\left\|\nabla u_j\right\|^2 + \frac{\alpha}{2}\sum_{i=1}^{j} \|u_i - u_{i-1}\|^2_{\mathrm{H}^1(\Omega)} \leqslant C_{\varepsilon_3} + C_{\varepsilon_4} + C_{\varepsilon_5} + (C_{\varepsilon_1} + C_{\varepsilon_2} + C_{\varepsilon_4})\sum_{i=1}^{j} \|\nabla u_i\|^2\, \tau.$$

We fix $(\varepsilon_i)_{i=1}^{5}$ sufficiently small such that $\varepsilon_5 < \frac{1}{4}$ and $\sum_{i=1}^{4} \varepsilon_i < \frac{k(T)}{4}$, and afterwards we apply Grönwall's lemma to conclude the proof. □

Remark 2. *In this paper, we have supposed that $f \in \mathrm{L}^\infty\left((0,T), \mathrm{L}^2(\Omega)\right)$. If $f \in \mathrm{L}^\infty\left((0,T), \mathrm{H}^1_0(\Omega)^*\right)$, then the term containing f on the right-hand side of (12) cannot be estimated directly. Instead, we can assume that $f \in \mathrm{H}^1\left((0,T), \mathrm{H}^1_0(\Omega)^*\right)$, since in that case we can estimate this term by using the following partial summation rule*

$$\sum_{i=1}^{j} \langle f_i, \delta u_i \rangle_{\mathrm{H}^1_0(\Omega)^* \times \mathrm{H}^1_0(\Omega)} \tau = \langle f_j, u_j \rangle_{\mathrm{H}^1_0(\Omega)^* \times \mathrm{H}^1_0(\Omega)} - \langle f_0, u_0 \rangle_{\mathrm{H}^1_0(\Omega)^* \times \mathrm{H}^1_0(\Omega)} - \sum_{i=1}^{j} \langle \delta f_i, u_{i-1} \rangle_{\mathrm{H}^1_0(\Omega)^* \times \mathrm{H}^1_0(\Omega)} \tau,$$

i.e.,

$$\left|\sum_{i=1}^{j}\langle f_i,\delta u_i\rangle_{\mathrm{H}_0^1(\Omega)^*\times\mathrm{H}_0^1(\Omega)}\tau\right|\leqslant C_\varepsilon+\varepsilon\left\|\nabla u_j\right\|^2+C\sum_{i=1}^{j-1}\|\nabla u_i\|^2\,\tau.$$

We note also that the condition on f in Theorem 1 cannot be relaxed to $f\in\mathrm{L}^\infty\left((0,T),\mathrm{L}^2(\Omega)\right)$ following the approach in (Reference [27] Lemma 3.3 and 3.4). However, it is clear that Theorem 1 is also satisfied when f is satisfying one of the conditions on f with the space $\mathrm{H}_0^1(\Omega)^*$ replaced by $\mathrm{L}^2(\Omega)$.

Corollary 1. *Let the assumptions of Lemma 2 be fulfilled. Then, there exist positive constants C such that for every $j=1,2,\ldots,n$, the following relation holds*

$$\left\|(k*\delta^2u)_j\right\|_{\mathrm{H}_0^1(\Omega)^*}\leqslant C.$$

Proof. The estimate follows from

$$\begin{aligned}\left\|(k*\delta^2u)_i\right\|_{\mathrm{H}_0^1(\Omega)^*}&=\sup_{\|\varphi\|_{\mathrm{H}_0^1(\Omega)}=1}\left|\langle(k*\delta^2u)_i,\varphi\rangle_{\mathrm{H}_0^1(\Omega)^*\times\mathrm{H}_0^1(\Omega)}\right|\\&=\sup_{\|\varphi\|_{\mathrm{H}_0^1(\Omega)}=1}\left|(f_i,\varphi)+(F(u_{i-1}),\varphi)-\mathcal{L}_i(u_i,\varphi)\right|\\&\leqslant\left\|f_i\right\|+C+C\left\|u_{i-1}\right\|+C\left\|\nabla u_i\right\|,\end{aligned}$$

and the result of Lemma 2. □

3.2. Convergence

In this subsection, the existence of a weak solution is proved using Rothe's method. We define the following piecewise linear in time functions $u_n:[0,T]\to\mathrm{L}^2(\Omega)$

$$u_n:[0,T]\to\mathrm{L}^2(\Omega):t\mapsto\begin{cases}\tilde{u}_0 & t=0\\ u_{i-1}+(t-t_{i-1})\delta u_i & t\in(t_{i-1},t_i],\quad 1\leqslant i\leqslant n,\end{cases}$$

and the piecewise constant in time functions $\overline{u}_n,\overline{v}_n,\tilde{u}_n:[0,T]\to\mathrm{L}^2(\Omega)$:

$$\overline{u}_n:[0,T]\to\mathrm{L}^2(\Omega):t\mapsto\begin{cases}\tilde{u}_0 & t=0\\ u_i & t\in(t_{i-1},t_i],\quad 1\leqslant i\leqslant n;\end{cases}$$

$$\overline{v}_n:[0,T]\to\mathrm{L}^2(\Omega):t\mapsto\begin{cases}\tilde{v}_0 & t=0\\ \delta u_i & t\in(t_{i-1},t_i],\quad 1\leqslant i\leqslant n;\end{cases}$$

$$\tilde{u}_n:[0,T]\to\mathrm{L}^2(\Omega):t\mapsto\begin{cases}\tilde{u}_0 & t\in[0,\tau]\\ \overline{u}_n(t-\tau) & t\in(t_{i-1},t_i],\quad 2\leqslant i\leqslant n.\end{cases}$$

Similarly, we define $\overline{k}_n$, $\overline{\mathcal{L}}_n$ and $\overline{f}_n$. We rewrite Equation (8) on the whole time frame by aid of these Rothe's functions and Equation (11) as follows

$$\begin{aligned}\langle\partial_t(k*\delta u)_n(t)-\overline{k}_n(t)\tilde{v}_0,\varphi\rangle_{\mathrm{H}_0^1(\Omega)^*\times\mathrm{H}_0^1(\Omega)}&+\overline{\mathcal{L}}_n(t)(\overline{u}_n(t),\varphi)\\&=\left(\overline{f}_n(t),\varphi\right)+(F(\tilde{u}_n(t)),\varphi),\quad\forall\varphi\in\mathrm{H}_0^1(\Omega).\end{aligned}\tag{13}$$

Note that

$$\overline{\mathcal{L}}_n(t)\left(\overline{u}_n(t),\varphi\right)=\left(\overline{a}_n(t)\nabla\overline{u}_n(t)+\overline{\boldsymbol{b}}_n(t)\overline{u}_n(t),\nabla\varphi\right)+\left(\overline{c}_n(t)\overline{u}_n(t),\varphi\right).$$

We show the existence of a unique weak solution in the following theorem.

Theorem 2 (Existence and uniqueness)**.** *Suppose that the conditions of Lemma 2 are fulfilled. Then, there exists a unique weak solution u to the problem (4) with $u\in \mathrm{C}\left([0,T],\mathrm{L}^2(\Omega)\right)\cap\mathrm{L}^\infty\left((0,T),\mathrm{H}^1_0(\Omega)\right)$ with $\partial_t u\in\mathrm{C}\left([0,T],\mathrm{H}^1_0(\Omega)^*\right)\cap\mathrm{L}^2\left((0,T),\mathrm{L}^2(\Omega)\right)$ and $k*\partial_{tt}u\in\mathrm{L}^2\left((0,T),\mathrm{H}^1_0(\Omega)^*\right)$.*

Proof. The compact embedding of $\mathrm{H}^1_0(\Omega)$ in $\mathrm{L}^2(\Omega)$ follows from the Rellich-Kondrachov theorem ([31] Theorem 6.6-3). Lemma 2 gives that the sequences $(u_n)_{n\in\mathbb{N}}$ is bounded in $\mathrm{L}^2\left((0,T),\mathrm{H}^1_0(\Omega)\right)$. Thus we have the existence of an element u in $\mathrm{L}^2\left((0,T),\mathrm{L}^2(\Omega)\right)$ and a subsequence $(u_{n_l})_{l\in\mathbb{N}}$ of $(u_n)_{n\in\mathbb{N}}$ such that

$$u_{n_l}\to u \text{ in } \mathrm{L}^2\left((0,T),\mathrm{L}^2(\Omega)\right) \text{ as } l\to\infty.$$

Moreover, the reflexivity of the space $\mathrm{L}^2\left((0,T),\mathrm{H}^1_0(\Omega)\right)$ implies the existence of a subsequence (indexed by n_l again) such that

$$u_{n_l}\rightharpoonup u \text{ in } \mathrm{L}^2\left((0,T),\mathrm{H}^1_0(\Omega)\right) \text{ as } l\to\infty.$$

Lemma 2 gives also that $u\in\mathrm{L}^\infty\left((0,T),\mathrm{H}^1_0(\Omega)\right)$ and that $(\partial_t u_{n_l}=\overline{v}_{n_l})_{l\in\mathbb{N}}$ is bounded in the reflexive space $\mathrm{L}^2\left((0,T),\mathrm{L}^2(\Omega)\right)$.

Therefore,

$$\overline{v}_{n_l}=\partial_t u_{n_l}\rightharpoonup\partial_t u \text{ in } \mathrm{L}^2\left((0,T),\mathrm{L}^2(\Omega)\right) \text{ as } l\to\infty. \tag{14}$$

Hence, $u\in\mathrm{C}\left([0,T],\mathrm{L}^2(\Omega)\right)$, see ([32], Lemma 7.3). Finally, from Lemma 2, it follows that

$$\int_0^T\left\|u_{n_l}(t)-\overline{u}_{n_l}(t)\right\|^2_{\mathrm{H}^1_0(\Omega)}\,\mathrm{d}t+\int_0^T\left\|\overline{u}_{n_l}(t)-\tilde{u}_{n_l}(t)\right\|^2_{\mathrm{H}^1_0(\Omega)}\,\mathrm{d}t\leqslant 2\tau_{n_l}\sum_{i=1}^{n_l}\|\nabla u_i-\nabla u_{i-1}\|^2\leqslant C\tau_{n_l},$$

that is, we have that

$$\overline{u}_{n_l},\tilde{u}_{n_l}\to u \text{ in } \mathrm{L}^2\left((0,T),\mathrm{L}^2(\Omega)\right) \text{ as } l\to\infty.$$

and

$$\overline{u}_{n_l}\rightharpoonup u \text{ in } \mathrm{L}^2\left((0,T),\mathrm{H}^1_0(\Omega)\right) \text{ as } l\to\infty.$$

Next, we integrate Equation (13) in time over $(0,\eta)\subset(0,T)$ for the resulting subsequence to obtain that

$$\begin{aligned}\langle(k*\delta u)_{n_l}(\eta),\varphi\rangle_{\mathrm{H}^1_0(\Omega)^*\times\mathrm{H}^1_0(\Omega)}&-\int_0^\eta\langle\tilde{v}_0\overline{k}_{n_l}(t),\varphi\rangle_{\mathrm{H}^1_0(\Omega)^*\times\mathrm{H}^1_0(\Omega)}\,\mathrm{d}t\\&+\int_0^\eta\overline{\mathcal{L}}_{n_l}(t)(\overline{u}_{n_l}(t),\varphi)\,\mathrm{d}t=\int_0^\eta\left(\overline{f}_{n_l}(t),\varphi\right)\mathrm{d}t+\int_0^\eta\left(F(\tilde{u}_{n_l}(t)),\varphi\right)\mathrm{d}t.\end{aligned}\tag{15}$$

We have that $\left\|\overline{\boldsymbol{A}}_n-\boldsymbol{A}\right\|_\infty\to 0$, $\left\|\overline{\boldsymbol{b}}_n-\boldsymbol{b}\right\|_\infty\to 0$ and $\left|\overline{c}_n-c\right|\to 0$ almost everywhere (a.e.) in Q_T as $n\to\infty$. Using the results above, we obtain for $\eta\in(0,T)$ that

$$\left|\int_0^\eta\overline{\mathcal{L}}_{n_l}(t)(\overline{u}_{n_l}(t),\varphi)\,\mathrm{d}t-\int_0^\eta\mathcal{L}(t)(u(t),\varphi)\,\mathrm{d}t\right|\to 0 \quad \text{as } n\to\infty,$$

$$\left|\int_0^\eta \left(\overline{f}_{n_l}(t),\varphi\right)\mathrm{d}t - \int_0^\eta (f,\varphi)\,\mathrm{d}t\right| \to 0 \quad \text{as } n\to\infty,$$
$$\left|\int_0^\eta \left(F(\tilde{u}_{n_l}(t)),\varphi\right)\mathrm{d}t - \int_0^\eta (F(u(t)),\varphi)\,\mathrm{d}t\right| \to 0 \quad \text{as } n\to\infty.$$

Now, we explain the limit transition in the first term of (15) by proving in two steps that

$$\lim_{l\to\infty}\left|\int_0^T \langle (k*\delta u)_{n_l}(t),\varphi\rangle_{\mathrm{H}_0^1(\Omega)^*\times \mathrm{H}_0^1(\Omega)}\,\mathrm{d}t - \int_0^T \langle (k*\overline{v}_{n_l})(t),\varphi\rangle_{\mathrm{H}_0^1(\Omega)^*\times \mathrm{H}_0^1(\Omega)}\,\mathrm{d}t\right| = 0. \tag{16}$$

These two steps are given by

$$\text{(i)}\ \lim_{l\to\infty}\left|\int_0^T \langle (k*\delta u)_{n_l}(t)\,\mathrm{d}t - \int_0^T \overline{(k*\delta u)}_{n_l}(t),\varphi\rangle_{\mathrm{H}_0^1(\Omega)^*\times \mathrm{H}_0^1(\Omega)}\,\mathrm{d}t\right| = 0;$$
$$\text{(ii)}\ \lim_{l\to\infty}\left|\int_0^T \langle\overline{(k*\delta u)}_{n_l}(t)\,\mathrm{d}t - \int_0^T (k*\overline{v}_{n_l})(t),\varphi\rangle_{\mathrm{H}_0^1(\Omega)^*\times \mathrm{H}_0^1(\Omega)}\,\mathrm{d}t\right| = 0.$$

From Corollary 1 and the Lebesgue dominated theorem, we obtain the limit transition (i) as follows

$$\begin{aligned}
&\int_0^T \left|\langle (k*\delta u)_{n_l}(t) - \overline{(k*\delta u)}_{n_l}(t),\varphi\rangle_{\mathrm{H}_0^1(\Omega)^*\times \mathrm{H}_0^1(\Omega)}\right|\mathrm{d}t\\
&= \sum_{i=1}^{n_l}\int_{t_{i-1}}^{t_i}\left|\langle (t-t_i)\delta(k*\delta u)_i,\varphi\rangle_{\mathrm{H}_0^1(\Omega)^*\times \mathrm{H}_0^1(\Omega)}\right|\mathrm{d}t\\
&\overset{(11)}{\leqslant} \sum_{i=1}^{n_l}\tau_{n_l}^2\left|\langle (k*\delta^2 u)_i - k_i\tilde{v}_0,\varphi\rangle_{\mathrm{H}_0^1(\Omega)^*\times \mathrm{H}_0^1(\Omega)}\right|\\
&\leqslant C\tau_{n_l} + C\int_1^2 \tau_{n_l}^{2-\beta}\mu(\beta)\,\mathrm{d}\beta \overset{l\to\infty}{\longrightarrow} 0.
\end{aligned}$$

For the limit transition (ii), we see that ($\lceil t\rceil_\tau = t_i$ when $t\in(t_{i-1},t_i]$)

$$\begin{aligned}
&\int_0^T \left|\langle\overline{(k*\delta u)}_{n_l}(t) - (k*\overline{v}_{n_l})(t),\varphi\rangle_{\mathrm{H}_0^1(\Omega)^*\times \mathrm{H}_0^1(\Omega)}\right|\mathrm{d}t\\
&= \int_0^T \left|\langle (\overline{k}_{n_l}*\overline{v}_{n_l})(\lceil t\rceil_\tau) - (k*\overline{v}_{n_l})(t),\varphi\rangle_{\mathrm{H}_0^1(\Omega)^*\times \mathrm{H}_0^1(\Omega)}\right|\mathrm{d}t\\
&\leqslant \int_0^T \left|\langle \int_0^t \left(\overline{k}_{n_l}(\lceil t\rceil_\tau - s) - k(t-s)\right)\overline{v}_{n_l}(s)\ \mathrm{d}s,\varphi\rangle_{\mathrm{H}_0^1(\Omega)^*\times \mathrm{H}_0^1(\Omega)}\right|\mathrm{d}t\\
&+ \int_0^T \left|\langle \int_t^{\lceil t\rceil_\tau} \overline{k}_{n_l}\left(\lceil t\rceil_\tau - s\right)\overline{v}_{n_l}(s)\ \mathrm{d}s,\varphi\rangle_{\mathrm{H}_0^1(\Omega)^*\times \mathrm{H}_0^1(\Omega)}\right|\mathrm{d}t.
\end{aligned}$$

We apply two times Hölder's inequality on the previous inequality and we obtain that

$$\begin{aligned}
&\int_0^T \left|\langle\overline{(k*\delta u)}_{n_l}(t) - (k*\overline{v}_{n_l})(t),\varphi\rangle_{\mathrm{H}_0^1(\Omega)^*\times \mathrm{H}_0^1(\Omega)}\right|\mathrm{d}t\\
&\leqslant \|\varphi\|_{\mathrm{H}_0^1(\Omega)}\left[\int_0^T\left(\int_0^t \left|\overline{k}_{n_l}\left(\lceil t\rceil_\tau - s\right) - k\left(t-s\right)\right|\ \mathrm{d}s\right)\mathrm{d}t\right]^{\frac{1}{2}}\\
&\times\left[\int_0^T\left(\int_0^t \left|\overline{k}_{n_l}(\lceil t\rceil_\tau - s) - k(t-s)\right|\left\|\overline{v}_{n_l}(s)\right\|^2_{\mathrm{H}_0^1(\Omega)^*}\ \mathrm{d}s\right)\mathrm{d}t\right]^{\frac{1}{2}}
\end{aligned}$$

$$+\left\|\varphi\right\|_{\mathrm{H}_0^1(\Omega)}\left[\int_0^T\left(\int_t^{\lceil t\rceil_\tau}\overline{k}_{n_l}\left(\lceil t\rceil_\tau-s\right)\,\mathrm{d}s\right)\mathrm{d}t\right]^{\frac{1}{2}}\left[\int_0^T\left(\int_t^{\lceil t\rceil_\tau}\overline{k}_{n_l}(\lceil t\rceil_\tau-s)\left\|\overline{v}_{n_l}(s)\right\|^2_{\mathrm{H}_0^1(\Omega)^*}\,\mathrm{d}s\right)\mathrm{d}t\right]^{\frac{1}{2}}.$$

From Lemma 2, it follows that (for $t\in(t_{i-1},t_i]$)

$$\int_t^{\lceil t\rceil_\tau}\overline{k}_{n_l}(\lceil t\rceil_\tau-s)\left\|\overline{v}_{n_l}(s)\right\|^2_{\mathrm{H}_0^1(\Omega)^*}\,\mathrm{d}s\leqslant C\int_0^{t_i}\overline{k}_{n_l}(t_i-s)\left\|\overline{v}_{n_l}(s)\right\|^2\,\mathrm{d}s=C\left(k*\|\delta u\|^2\right)_i\leqslant C,$$

and

$$\int_0^T\left(\int_0^t\left|\overline{k}_{n_l}(\lceil t\rceil_\tau-s)-k(t-s)\right|\left\|\overline{v}_{n_l}(s)\right\|^2_{\mathrm{H}_0^1(\Omega)^*}\,\mathrm{d}s\right)\mathrm{d}t$$
$$\overset{(\star)}{\leqslant}\|k\|_{\mathrm{L}^1(0,T)}\left\|\overline{v}_{n_l}\right\|_{\mathrm{L}^2\left((0,T),\mathrm{H}_0^1(\Omega)^*\right)}+\int_0^T\left(\int_0^t\overline{k}_{n_l}\left(\lceil t\rceil_\tau-s\right)\left\|\overline{v}_{n_l}(s)\right\|^2_{\mathrm{H}_0^1(\Omega)^*}\,\mathrm{d}s\right)\mathrm{d}t\leqslant C,$$

where we employed Young's inequality for convolutions at position $(\star)$. Moreover, for $t\in(t_{i-1},t_i]$, by the Lebesgue dominated theorem, we obtain that

$$\int_t^{\lceil t\rceil_\tau}\overline{k}_{n_l}\left(\lceil t\rceil_\tau-s\right)\,\mathrm{d}s=\int_0^{t_i-t}\left(\int_1^2\frac{\tau_{n_l}^{1-\beta}}{\Gamma(2-\beta)}\mu(\beta)\,\mathrm{d}\beta\right)\mathrm{d}\xi\leqslant\int_1^2\tau_{n_l}^{2-\beta}\mu(\beta)\,\mathrm{d}\beta\to0\quad\text{as }\tau_{n_l}\to0,$$

since $\Gamma(z)\geqslant\Gamma(1)=1$ for all $z\in(0,1)$. We also have for $t\in(t_{i-1},t_i]$ that

$$\begin{aligned}
&\int_0^t\left|\overline{k}_{n_l}\left(\lceil t\rceil_\tau-s\right)-\overline{k}_{n_l}\left(t-s\right)\right|\,\mathrm{d}s\\
&=\int_0^{t_{i-1}}\left(\overline{k}_{n_l}\left(t-s\right)-\overline{k}_{n_l}\left(t_i-s\right)\right)\,\mathrm{d}s+\int_{t_{i-1}}^t\left(\overline{k}_{n_l}\left(t-s\right)-\overline{k}_{n_l}\left(t_i-s\right)\right)\,\mathrm{d}s\\
&\leqslant\int_0^{t_{i-1}}\left(\overline{k}_{n_l}\left(t_{i-1}-s\right)-\overline{k}_{n_l}\left(t_i-s\right)\right)\,\mathrm{d}s+2\int_{t_{i-1}}^t\overline{k}_{n_l}\left(t-s\right)\,\mathrm{d}s\\
&\leqslant\int_0^{t_{i-1}}\left(k\left(t_{i-1}-s\right)-k\left(t_{i+1}-s\right)\right)\,\mathrm{d}s+2\int_{t_{i-1}}^t k\left(t-s\right)\,\mathrm{d}s\\
&\leqslant\int_1^2\left(\frac{t_{i-1}^{2-\beta}}{\Gamma(3-\beta)}+\frac{(2\tau_{n_l})^{2-\beta}}{\Gamma(3-\beta)}-\frac{t_{i+1}^{2-\beta}}{\Gamma(3-\beta)}+2\frac{(t-t_{i-1})^{2-\beta}}{\Gamma(3-\beta)}\right)\mu(\beta)\,\mathrm{d}\beta\\
&\leqslant C\int_1^2\tau_{n_l}^{2-\beta}\mu(\beta)\,\mathrm{d}\beta\to0\quad\text{as }\tau_{n_l}\to0,
\end{aligned}$$

using $\overline{k}_{n_l}\leqslant k$ (since k is decreasing in time), $\Gamma(z)\geqslant\frac{1}{2}$ for $z\in(1,2)$ and the α-Hölder continuity of $f(x)=x^\alpha$ when $\alpha\in(0,1)$. In a similar way, we get that

$$\int_0^t\left|\overline{k}_{n_l}\left(t-s\right)-k\left(t-s\right)\right|\,\mathrm{d}s\leqslant C\int_1^2\tau_{n_l}^{2-\beta}\mu(\beta)\,\mathrm{d}\beta\to0\quad\text{as }\tau_{n_l}\to0$$

and thus

$$\int_0^t\left|\overline{k}_{n_l}\left(\lceil t\rceil_\tau-s\right)-k\left(t-s\right)\right|\,\mathrm{d}s\leqslant C\int_1^2\tau_{n_l}^{2-\beta}\mu(\beta)\,\mathrm{d}\beta\to0\quad\text{as }\tau_{n_l}\to0.$$

Hence, the limit transition (ii) is valid. Now, we integrate Equation (15) again in time over $\eta\in(0,\xi)\subset(0,T)$, that is,

$$\int_0^\xi\langle(k*\delta u)_{n_l}(\eta),\varphi\rangle_{\mathrm{H}_0^1(\Omega)^*\times\mathrm{H}_0^1(\Omega)}\,\mathrm{d}\eta-\int_0^\xi\int_0^\eta\langle\tilde{v}_0\overline{k}_{n_l}(t),\varphi\rangle_{\mathrm{H}_0^1(\Omega)^*\times\mathrm{H}_0^1(\Omega)}\,\mathrm{d}t\,\mathrm{d}\eta$$

$$+\int_0^{\xi}\int_0^{\eta}\overline{\mathcal{L}}_{n_l}(t)(\overline{u}_{n_l}(t),\varphi)\,\mathrm{d}t\,\mathrm{d}\eta=\int_0^{\xi}\int_0^{\eta}\left(\overline{f}_{n_l}(t),\varphi\right)\mathrm{d}t\,\mathrm{d}\eta+\int_0^{\xi}\int_0^{\eta}\left(F(\tilde{u}_{n_l}(t)),\varphi\right)\mathrm{d}t\,\mathrm{d}\eta. \quad (17)$$

Then, since the following integral represents a linear bounded functional on the space $\mathrm{L}^2\left((0,T),\mathrm{L}^2(\Omega)\right)$:

$$\begin{aligned}\left|\int_0^{\eta}\langle(k*\overline{v}_{n_l})(t),\varphi\rangle_{\mathrm{H}_0^1(\Omega)^*\times\mathrm{H}_0^1(\Omega)}\,\mathrm{d}t\right| &\leqslant \|\varphi\|_{\mathrm{H}_0^1(\Omega)}\int_0^{\eta}\left(k*\|\overline{v}_{n_l}\|_{\mathrm{H}_0^1(\Omega)^*}\right)(t)\,\mathrm{d}t\\ &\leqslant \|\varphi\|\,\|k\|_{\mathrm{L}^1(0,\eta)}\left\|\|\overline{v}_{n_l}\|_{\mathrm{H}_0^1(\Omega)^*}\right\|_{\mathrm{L}^1(0,\eta)}\\ &\leqslant C\|\varphi\|\,\|k\|_{\mathrm{L}^1(0,T)}\|\overline{v}_{n_l}\|_{\mathrm{L}^2\left((0,T),\mathrm{L}^2(\Omega)\right)},\end{aligned}$$

using Equations (14) and (16), we can pass to the limit for $l\to\infty$ in (17). We obtain that (as also $\overline{k}_{n_l}\to k$ pointwise in $(0,T)$)

$$\begin{aligned}&\int_0^{\xi}\langle(k*\partial_t u)(\eta),\varphi\rangle_{\mathrm{H}_0^1(\Omega)^*\times\mathrm{H}_0^1(\Omega)}\,\mathrm{d}\eta-\int_0^{\xi}\int_0^{\eta}\langle\tilde{v}_0k(t),\varphi\rangle_{\mathrm{H}_0^1(\Omega)^*\times\mathrm{H}_0^1(\Omega)}\,\mathrm{d}t\,\mathrm{d}\eta\\ &\quad+\int_0^{\xi}\int_0^{\eta}\mathcal{L}(t)(u(t),\varphi)\,\mathrm{d}t\,\mathrm{d}\eta=\int_0^{\xi}\int_0^{\eta}(f(t),\varphi)\,\mathrm{d}t\,\mathrm{d}\eta+\int_0^{\xi}\int_0^{\eta}(F(u(t)),\varphi)\,\mathrm{d}t\,\mathrm{d}\eta. \quad (18)\end{aligned}$$

We differentiate this relation with respect to ξ, that is,

$$\begin{aligned}&\langle(k*\partial_t u)(\xi),\varphi\rangle_{\mathrm{H}_0^1(\Omega)^*\times\mathrm{H}_0^1(\Omega)}-\int_0^{\xi}\langle\tilde{v}_0k(t),\varphi\rangle_{\mathrm{H}_0^1(\Omega)^*\times\mathrm{H}_0^1(\Omega)}\,\mathrm{d}t+\int_0^{\xi}\mathcal{L}(t)(u(t),\varphi)\,\mathrm{d}t\\ &\qquad=\int_0^{\xi}(f(t),\varphi)\,\mathrm{d}t+\int_0^{\xi}(F(u(t)),\varphi)\,\mathrm{d}t. \quad (19)\end{aligned}$$

Hence, $(k*\partial_t u)(t)\in\mathrm{H}_0^1(\Omega)^*$ for all $t\in(0,T)$. Moreover, since $u\in\mathrm{L}^\infty\left((0,T),\mathrm{H}_0^1(\Omega)\right)$ and $f\in\mathrm{L}^\infty\left((0,T),\mathrm{L}^2(\Omega)\right)$, we have that

$$\lim_{\xi\overset{>}{\to}0}\langle(k*\partial_t u)(\xi),\varphi\rangle_{\mathrm{H}_0^1(\Omega)^*\times\mathrm{H}_0^1(\Omega)}=0\quad\Rightarrow\quad(k*\partial_t u)(0)=0\text{ in }\mathrm{H}_0^1(\Omega)^*.$$

Next, differentiating (19) again with respect to ξ (and afterwards replacing ξ by t) gives

$$\begin{aligned}&\langle\partial_t(k*\partial_t u)(t)-k(t)\tilde{v}_0,\varphi\rangle_{\mathrm{H}_0^1(\Omega)^*\times\mathrm{H}_0^1(\Omega)}+\mathcal{L}(t)\,(u(t),\varphi)\\ &\qquad=(f(t),\varphi)+(F(u(t)),\varphi),\qquad\forall\varphi\in\mathrm{H}_0^1(\Omega). \quad (20)\end{aligned}$$

Thus u is satisfying (4) since $\partial_t(k*\partial_t u)-k\tilde{v}_0=k*\partial_{tt}u\in\mathrm{L}^2\left((0,T),\mathrm{H}_0^1(\Omega)^*\right)$ as $\partial_t u$ yields to be absolutely continuous. We integrate (20) in time and obtain that $(k*(\partial_t u-\tilde{v}_0))(t)$ is absolutely continuous with values in $\mathrm{H}_0^1(\Omega)^*$ as $\partial_t(k*\partial_t u)(t)-k(t)\tilde{v}_0=\partial_t\left(k*(\partial_t u-\tilde{v}_0)\right)(t)$ in $\mathrm{H}_0^1(\Omega)^*$. Moreover, there exists a non-negative kernel $g\in\mathrm{L}^1(0,T)$ such that $g*k=1$ ([28] Proposition 6), and thus

$$(g*\partial_t\left(k*(\partial_t u-\tilde{v}_0)\right))(t)=\partial_t\left(g*k*(\partial_t u-\tilde{v}_0)\right)(t)=\partial_t u(t)-\tilde{v}_0\quad\text{ in }\mathrm{H}_0^1(\Omega)^*,$$

as $\left(k*(\partial_t u-\tilde{v}_0)\right)(0)=0$. Therefore, applying this convolution operation on (20) gives that $\partial_t u$ is absolutely continuous in the time variable, that is, $\partial_t u\in\mathrm{C}\left([0,T],\mathrm{H}_0^1(\Omega)^*\right)$.

We finish the proof by showing the uniqueness of a solution. The proof is by contradiction. We suppose that two solutions u_1 and u_2 to (4) exist and we set $u=u_1-u_2$. Then $u(\cdot,0)=0$

and $\partial_t u(\cdot,0) = 0$ in Ω. We cannot choose $\varphi = \partial_t u(t)$ in (4) as $\partial_t u(t) \notin \mathrm{H}_0^1(\Omega)$. Instead, we put $\varphi = u(\eta) \in \mathrm{H}_0^1(\Omega)$ in (18). First note that

$$\partial_t (k * u)(\mathbf{x},t) = (k * \partial_t u)(\mathbf{x},t) + k(t)u(\mathbf{x},0) = (k * \partial_t u)(\mathbf{x},t), \qquad (\mathbf{x},t) \in Q_T.$$

Then, we get that the difference u is satisfying

$$\int_0^\xi \langle \partial_t (k * u)(\eta), u(\eta)\rangle_{\mathrm{H}_0^1(\Omega)^* \times \mathrm{H}_0^1(\Omega)} \,\mathrm{d}\eta + \int_0^\xi \int_0^\eta \mathcal{L}(t)(u(t),u(\eta))\,\mathrm{d}t\,\mathrm{d}\eta$$
$$= \int_0^\xi \left(\int_0^\eta [F(u_1(t)) - F(u_2(t))]\,\mathrm{d}t, u(\eta) \right) \mathrm{d}\eta. \quad (21)$$

As $\frac{t^{1-\beta}}{\Gamma(2-\beta)} = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}$ with $\alpha = \beta - 1 \in (0,1)$, we can apply ([33] Proposition 10) and we obtain for the first term on the LHS of (21) that

$$\begin{aligned}
\int_0^\xi \langle \partial_t (k * u)(\eta), u(\eta)\rangle_{\mathrm{H}_0^1(\Omega)^* \times \mathrm{H}_0^1(\Omega)} \,\mathrm{d}\eta &= \int_\Omega \int_1^2 \mu(\beta) \int_0^\xi \partial_t \left(\frac{t^{1-\beta}}{\Gamma(2-\beta)} * u(\mathbf{x}) \right)(\eta) u(\mathbf{x},\eta)\,\mathrm{d}\eta\,\mathrm{d}\beta\ \mathrm{d}\mathbf{x} \\
&\geqslant \int_\Omega \int_1^2 \mu(\beta) \frac{\xi^{1-\beta}}{2\Gamma(2-\beta)} \int_0^\xi |u(\mathbf{x},\eta)|^2 \,\mathrm{d}\eta\,\mathrm{d}\beta\ \mathrm{d}\mathbf{x} \\
&\geqslant \frac{k(T)}{2} \int_0^\xi \|u(\eta)\|^2 \,\mathrm{d}\eta.
\end{aligned}$$

The analysis of the second term on the LHS of (21) is split into multiple parts. Employing multiple times integration by parts and the symmetry of $\boldsymbol{A}$, we get that

$$\begin{aligned}
&\int_0^\xi \left(\int_0^\eta \boldsymbol{A}(t)\nabla u(t)\,\mathrm{d}t, \nabla u(\eta) \right) \mathrm{d}\eta \\
&= - \int_0^\xi \left(\nabla u(\eta), \int_0^\eta \boldsymbol{A}(t)\nabla u(t)\,\mathrm{d}t + \int_0^\eta \partial_t \boldsymbol{A}(t) \left[\int_0^t \nabla u(s)\ \mathrm{d}s \right] \mathrm{d}t \right) \mathrm{d}\eta \\
&\quad - \left(\int_0^\xi \partial_t \boldsymbol{A}(t) \left[\int_0^t \nabla u(s)\ \mathrm{d}s \right] \mathrm{d}t, \int_0^\xi \nabla u(t)\,\mathrm{d}t \right) + \left(\boldsymbol{A}(\xi) \left[\int_0^\xi \nabla u(s)\ \mathrm{d}s \right], \int_0^\xi \nabla u(t)\,\mathrm{d}t \right)
\end{aligned}$$

and thus

$$\begin{aligned}
\int_0^\xi \left(\int_0^\eta \boldsymbol{A}(t)\nabla u(t)\,\mathrm{d}t, \nabla u(\eta) \right) \mathrm{d}\eta &= \frac{1}{2} \left(\boldsymbol{A}(\xi) \left[\int_0^\xi \nabla u(s)\ \mathrm{d}s \right], \int_0^\xi \nabla u(t)\,\mathrm{d}t \right) \\
+ \frac{1}{2} \int_0^\xi \left(\int_0^\eta \nabla u(t)\,\mathrm{d}t, \partial_t \boldsymbol{A}(\eta) \left[\int_0^\eta \nabla u(s)\ \mathrm{d}s \right] \right) \mathrm{d}\eta &- \left(\int_0^\xi \partial_t \boldsymbol{A}(t) \left[\int_0^t \nabla u(s)\ \mathrm{d}s \right] \mathrm{d}t, \int_0^\xi \nabla u(t)\,\mathrm{d}t \right).
\end{aligned}$$

Therefore, using the ε-Young inequality, we finally have that

$$\int_0^\xi \left(\int_0^\eta \boldsymbol{A}(t)\nabla u(t)\,\mathrm{d}t, \nabla u(\eta) \right) \mathrm{d}\eta \geqslant \left(\frac{\alpha}{2} - \varepsilon_1 \right) \left\| \int_0^\xi \nabla u(t)\,\mathrm{d}t \right\|^2 - C_{\varepsilon_1} \int_0^\xi \left\| \int_0^\eta \nabla u(t)\,\mathrm{d}t \right\|^2 \mathrm{d}\eta.$$

Moreover, using $\partial_t \boldsymbol{b} \in \left(\mathrm{L}^\infty \left(\overline{Q_T} \right) \right)^d$, we have that

$$\begin{aligned}
&\left| \int_0^\xi \left(\int_0^\eta \boldsymbol{b}(t)u(t)\,\mathrm{d}t, \nabla u(\eta) \right) \mathrm{d}\eta \right| \\
&= \left| \int_0^\xi \left(\int_0^\eta [(\nabla \cdot \boldsymbol{b})(t)u(t) + \boldsymbol{b}(t) \cdot \nabla u(t)]\,\mathrm{d}t, u(\eta) \right) \mathrm{d}\eta \right|
\end{aligned}$$

$$
\begin{aligned}
&\leqslant C_{\varepsilon_2}\int_0^{\xi}\left(\int_0^{\eta}\|u(t)\|^2\,\mathrm{d}t\right)\mathrm{d}\eta+\varepsilon_2\int_0^{\xi}\|u(\eta)\|^2\,\mathrm{d}\eta\\
&\quad+\left|\int_0^{\xi}\left(-\int_0^{\eta}\left[\partial_t\boldsymbol{b}(t)\cdot\left(\int_0^{t}\nabla u(s)\ \mathrm{d}s\right)\right]\mathrm{d}t+\boldsymbol{b}(\eta)\cdot\left(\int_0^{\eta}\nabla u(s)\ \mathrm{d}s\right),u(\eta)\right)\mathrm{d}\eta\right|\\
&\leqslant C_{\varepsilon_2}\int_0^{\xi}\left(\int_0^{\eta}\|u(t)\|^2\,\mathrm{d}t\right)\mathrm{d}\eta+(\varepsilon_2+\varepsilon_3+\varepsilon_4)\int_0^{\xi}\|u(\eta)\|^2\,\mathrm{d}\eta+(C_{\varepsilon_3}+C_{\varepsilon_4})\int_0^{\xi}\left\|\int_0^{\eta}\nabla u(s)\ \mathrm{d}s\right\|^2\mathrm{d}\eta
\end{aligned}
$$

and

$$
\left|\int_0^{\xi}\left(\int_0^{\eta}c(t)u(t)\,\mathrm{d}t,u(\eta)\right)\mathrm{d}\eta\right|\leqslant C_{\varepsilon_5}\int_0^{\xi}\left(\int_0^{\eta}\|u(t)\|^2\,\mathrm{d}t\right)\mathrm{d}\eta+\varepsilon_5\int_0^{\xi}\|u(\eta)\|^2\,\mathrm{d}\eta.
$$

From the Lipschitz continuity of F, we have that

$$
\left|\int_0^{\xi}\left(\int_0^{\eta}[F(u_1(t))-F(u_2(t))]\,\mathrm{d}t,u(\eta)\right)\mathrm{d}\eta\right|\leqslant C_{\varepsilon_6}\int_0^{\xi}\left(\int_0^{\eta}\|u(t)\|^2\,\mathrm{d}t\right)\mathrm{d}\eta+\varepsilon_6\int_0^{\xi}\|u(\eta)\|^2\,\mathrm{d}\eta.
$$

Collecting all these calculations above, we obtain from (21) that

$$
\begin{aligned}
&\left(\frac{k(T)}{2}-\sum_{i=2}^{6}\varepsilon_i\right)\int_0^{\xi}\|u(\eta)\|^2\,\mathrm{d}\eta+\left(\frac{\alpha}{2}-\varepsilon_1\right)\left\|\int_0^{\xi}\nabla u(t)\,\mathrm{d}t\right\|^2\\
&\qquad\leqslant(C_{\varepsilon_2}+C_{\varepsilon_5}+C_{\varepsilon_6})\int_0^{\xi}\left(\int_0^{\eta}\|u(t)\|^2\,\mathrm{d}t\right)\mathrm{d}\eta+(C_{\varepsilon_1}+C_{\varepsilon_3}+C_{\varepsilon_4})\int_0^{\xi}\left\|\int_0^{\eta}\nabla u(s)\ \mathrm{d}s\right\|^2\mathrm{d}\eta.
\end{aligned}
$$

First, we fix $\sum_{i=2}^{6}\varepsilon_i>0$ and $\varepsilon_1>0$ sufficiently small such that $\sum_{i=2}^{6}<\frac{k(T)}{2}$ and $\varepsilon_1<\frac{\alpha}{2}$. Afterwards, we apply Grönwall's lemma to obtain that $u=0$ a.e. in Q_T, that is, $u_1=u_2$ a.e. in Q_T. We note that the convergence of Rothe's functions towards the weak solution is also valid for the entire Rothe's sequence since the solution is unique. □

The result of Theorem 2 is also satisfied for the fractional wave equation itself, that is, when $k(t)=\frac{t^{1-\beta}}{\Gamma(2-\beta)}$. This can be seen by considering for example, that the $\int_1^2\tilde{f}(\tau_{n_l},\beta)\mu(\beta)\,\mathrm{d}\beta$ integrals in the proof become $\tilde{f}(\tau_{n_l},\beta)$, and also that $g(t)=\frac{t^{\beta-2}}{\Gamma(\beta-1)}$ in this case.

We conclude the paper by summarizing the results from this paper in the following theorem. Future work can concern to investigate the existence of a solution in the case that the initial data $\tilde{u}_0$ and $\tilde{v}_0$ belong to $\mathrm{L}^2(\Omega)$.

Theorem 3. *Consider* $\left(\partial_t^{\beta}u\right)(t)=(k*\partial_{tt}u)\,(t)=\int_0^t\frac{(t-s)^{1-\beta}}{\Gamma(2-\beta)}\partial_{ss}u(s)\ \mathrm{d}s$ *with* $\beta\in(1,2)$ *or* $\left(\mathcal{D}_t^{(\mu)}u\right)(t)=(k*\partial_{tt}u)\,(t)=\int_1^2\left(\partial_t^{\beta}u\right)(t)\mu(\beta)\,\mathrm{d}\beta$ *(i.e., DO) in problem (1). Assume that*

- $\mu\in\mathrm{L}^1(1,2),\mu\geqslant 0,\mu\not\equiv 0$ *(if DO);*
- $\tilde{u}_0\in\mathrm{H}_0^1(\Omega)$ *and* $\tilde{v}_0\in\mathrm{L}^2(\Omega)$*;*
- $f\in\mathrm{L}^{\infty}\left((0,T),\mathrm{L}^2(\Omega)\right)$ *or* $f\in\mathrm{H}^1\left((0,T),\mathrm{H}_0^1(\Omega)^*\right)$*;*
- $\boldsymbol{A}\in\left(\mathrm{L}^{\infty}\left(\overline{Q_T}\right)\right)^{d\times d}$ *is uniformly elliptic with ellipticity constant* α, $\boldsymbol{A}^{\top}=\boldsymbol{A}$ *and* $\partial_t\boldsymbol{A}\in\left(\mathrm{L}^{\infty}\left(\overline{Q_T}\right)\right)^{d\times d}$*;*
- $\boldsymbol{b}\in\mathbf{L}^{\infty}\left(\overline{Q_T}\right)$ *with* $\partial_t\boldsymbol{b}\in\left(\mathrm{L}^{\infty}\left(\overline{Q_T}\right)\right)^{d}$ *and* $(\nabla\cdot\boldsymbol{b})(t)\in\mathrm{L}^{\infty}(\Omega)$ *for all* $t\in(0,T)$*;*
- $c\in\mathrm{L}^{\infty}\left(\overline{Q_T}\right)$ *such that* $c\geqslant\frac{\|\boldsymbol{b}\|^2_{\mathrm{L}^{\infty}(\overline{Q_T})}}{2\alpha}$*;*
- $F:\mathbb{R}\to\mathbb{R}$ *is Lipschitz continuous.*

Then, there exists a unique weak solution u *to the problem (1) with* $u\in\mathrm{C}\left([0,T],\mathrm{L}^2(\Omega)\right)\cap\mathrm{L}^{\infty}\left((0,T),\mathrm{H}_0^1(\Omega)\right)$ *with* $\partial_t u\in\mathrm{C}\left([0,T],\mathrm{H}_0^1(\Omega)^*\right)\cap\mathrm{L}^2\left((0,T),\mathrm{L}^2(\Omega)\right)$ *and* $k*\partial_{tt}u\in\mathrm{L}^2\left((0,T),\mathrm{H}_0^1(\Omega)^*\right)$*.*

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Abbreviations

The following abbreviations are used in this manuscript:

a.a.	almost all
a.e.	almost everywhere
DO	distributed-order
LHS	left-hand side
RHS	right-hand side
PDE	partial differential equation

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